Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 11, NUMBER 1 (2008) ISSN 1311-0454

REPRINTED PAPER * by M. CAPUTO, from *Geophys. J. R. Astr. Soc.*, vol. 13, Issue 5 (1967), pp. 529-539

EDITORIAL NOTE: Nowadays, there are several basic definitions for the operators of differentiation and integration of fractional (arbitrary) order, as objects of the classical Fractional Calculus (FC). One of them is commonly known as the *Caputo derivative*. Appearing as a strange theoretical analogue of the conventional Calculus, with a very long and controversial history, FC has been recently recognized both by pure mathematicians and applied scientists and engineers, as an efficient and irreplaceable tool in modeling phenomena of the real world. One of its important and first applications in mechanics, especially to linear models of viscoelasticity, is due to Prof. Michele Caputo. In 1967, in his paper *"Linear Models of Dissipation whose Q is almost Frequency Independent – II"*, he introduced an operator for differentiation of arbitrary (fractional) order of the form, today referred to as the "Caputo derivative" (see eq. (5), p. 530) and used it for finding the analytical expression for a linear dissipative mechanism whose Q is almost frequency independent over large frequency ranges.

Prof. Caputo is a Honorary Member of Editorial Board of our journal. The year 2007 marked both his 80th personal anniversary, as well as *the 40th anniversary of this already famous paper*, and of the "Caputo derivative". On this occasion, in vol. 10 (2007), and especially No 3, we have invited several survey and contributed papers close to the subject of the Caputo paper'1967 and continuing its applications' trends. We find it now most interesting and useful for our audience, to reproduce the paper in this issue.

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* *Special thanks* are due, for permission to reprint this article, to: *Blackwell Publishing Ltd*, Publisher of *"Geophysical Journal International"* (GJI), as a heir to *"Geophysical Journal Royal Astronomical Society"*, 1967. In its original variant, the paper appeared in: *Geophys. J. R. Astr. Soc.*, vol. **13**, Issue 5, November 1967, pp. 529-539. Now, it can be found and ordered at GJI contents: http://www.blackwell-synergy.com/toc/gji/13/5.

ON BEHALF OF THE EDITORIAL BOARD OF "FCAA", Virginia Kiryakova

Geophys. J. R. astr. Soc. (1967) 13, 529–539.

Linear Models of Dissipation whose Q is almost Frequency Independent—II

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(Received 1967 May 1)

Summary

Laboratory experiments and field observations indicate that the Q of many non-ferromagnetic inorganic solids is almost frequency independent in the range 10^{-2}–10^{7} c/s, although no single substance has been investigated over the entire frequency spectrum. One of the purposes of this investigation is to find the analytic expression for a linear dissipative mechanism whose Q is almost frequency independent over large frequency ranges. This will be obtained by introducing fractional derivatives in the stress–strain relation.

Since the aim of this research is also to contribute to elucidating the dissipating mechanism in the Earth free modes, we shall treat the dissipation in the free, purely torsional, modes of a shell. The dissipation in a plane wave will also be treated.

The theory is checked with the new values determined for the Q of spheroidal free modes of the Earth in the range between 10 and 5 min integrated with the Q of Rayleigh waves in the range between 5 and 0·6 min.

Another check of the theory is made with the experimental values of the Q of the longitudinal waves in an aluminium rod in the range between 10^{-5} and 10^{-3} s.

In both checks the theory represents the observed phenomena very satisfactorily. The time derivative which enters the stress–strain relation in both cases is of order 0·15.

The present paper is a generalized version of another (Caputo 1966b) in which an elementary definition of some differential operators was used. In this paper we give also a rigorous proof of the formulae to be used in obtaining the analytic expression of Q; moreover, we present two checks of the theory with experimental data.

1. Introduction

The present paper is a generalized version of another (Caputo 1966b) in which an elementary definition of some differential operators was used. In this paper we give also a rigorous proof of the formulae to be used in obtaining the analytic expression of Q; moreover, we present two checks of the theory with experimental data.

In a homogeneous isotropic elastic field the elastic properties of the substance are specified by a description of the strains and stresses in a limited portion of the field since the strains and stresses are linearly related by two parameters which describe the elastic properties of the field. If the elastic field is not homogeneous nor isotropic the properties of the field are specified in a similar manner by a larger number of parameters which also depend on the position.

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These perfectly elastic fields are insufficient models for the description of many physical phenomena because they do not allow us to explain the dissipation of energy. A more complete description of the actual elastic fields is obtained by introducing stress–strain relations which include also linear combinations of time derivatives of the strain and the stress. The numerical coefficients appearing in the general linear combinations of higher order derivatives are called viscoelastic constants of higher order.

Elastic fields described by elastic constants of higher order have been discussed by many authors (e.g. see Knopoff (1954), Caputo (1966a). For a description of the various models of losses see Knopoff (1964).) Knopoff studied the case in which the stress–strain relations are of the type

$$\tau_{rs}=\lambda g^{hi} g_{rs} e_{hi}+2\mu e_{rs}+\frac{d^m}{dt^m}\left[\lambda_m g^{hi} g_{rs} e_{hi}+2\mu_m e_{rs}\right], \tag{1}$$

where λ_m and μ_m are constant; he obtained a condition for these viscoelastic constants of higher order analogous to those existing for the perfectly elastic fields and also proved that in order to have a dissipative elastic field the stress–strain relations should contain a time derivative of odd order.

A generalization of the relation (1) is

$$\tau_{rs}=\sum_{m=0}^{p}\frac{d^m}{dt^m}\left[\lambda_m g^{hi} g_{rs} e_{hi}+2\mu_m e_{rs}\right], \tag{2}$$

where one can also consider λ_m and μ_m as functions of position.

We can generalize (2) to the case when the operation d^m/dt^m is performed with m as a real number z and also further by replacing the summation with an integral as follows:

$$\tau_{rs}=\int_{a_1}^{b_1} f_1(r,z)\frac{d^z}{dt^z}\left(g^{hi} g_{rs} e_{hi}\right)dz+2\int_{a_2}^{b_2} f_2(r,z)\frac{d^z}{dt^z}\left(e_{rs}\right)dz. \tag{3}$$

Relations (1) and (2) are a special case of (3). They are obtained by setting

$$\left.\begin{aligned} f_1(r,z)&=\sum_{m=q}^{p}\delta(z-m)\,\lambda_m,\\ f_2(r,z)&=\sum_{m=q}^{p}\delta(z-m)\,\mu_m,\end{aligned}\right\} \tag{4}$$

where $\delta(z-m)$ are unitary delta functions.

If $a_i=p=q=0$ then we have the case of a perfectly elastic field; if $p=q=1$ then we have a perfectly viscous field; if $a_i=q=0$ and $p=1$ then we have a viscoelastic field.

We have now to establish a few relations which we shall have to use later.

Let $f(t)$ and its ith order derivatives $(i=1, 2, \ldots, m+1)$ be continuous in the interval $(0, +\infty)$ and also let z be a real number $(0<z<1)$.

We shall assume the definition of the derivative of $f(t)$ of order $m+z$ as follows:

$$\frac{d^{m+z}}{dt^{m+z}}f(t)=\frac{1}{\Gamma(1-z)}\int_0^t (t-\xi)^{-z} f^{(m+1)}(\xi)\,d\xi. \tag{5}$$

We want to prove that if $|f^{(i+1)}(t)|\,e^{-pt}$, $(p>0)$ $(i=0, 1, \ldots, m)$, is integrable in $(0, +\infty)$, then

$$\int_0^\infty\left[\frac{d^{m+z}}{dt^{m+z}}f(t)\right]e^{-pt}dt=p^z\left\{p^m\int_0^\infty f(\xi)\,e^{-p\xi}d\xi-p^{m-1}f(0)-p^{m-2}\ldots f(0)\ldots\right.$$
$$\left.\ldots-f^{(m-1)}(0)+p^{-1}f^{(m)}(0)\right\}. \tag{6}$$

From the definition (5) we have obviously

$$\int_0^\infty \left[\frac{d^{m+z}}{dt^{m+z}} f(t)\right] e^{-pt} dt = \frac{1}{\Gamma(1-z)} \int_0^\infty \left\{ \int_0^t (t-\xi)^{-z} f^{(m+1)}(\xi) d\xi \right\} e^{-pt} dt. \tag{7}$$

To prove (6) we need to change the order of integration in (7); by the theorem of Fubini-Tonelli this is possible if the function

$$|f^{(m+1)}(\xi)| \int_\xi^\infty |(t-\xi)^{-z} e^{-pt}| dt = |f^{(m+1)}(\xi)| \int_\xi^\infty (t-\xi)^{-z} e^{-pt} dt$$

is integrable in $(0, +\infty)$. This can be proved true as follows:

$$|f^{(m+1)}(\xi)| \int_\xi^\infty (t-\xi)^{-z} e^{-pt} dt = |f^{(m+1)}(\xi)| e^{-pq} p^{z-1} \Gamma(1-z)$$

which is integrable in $(0, +\infty)$ because of the hypothesis that

$$|f^{(m+1)}(\xi)| e^{-pq}$$

is integrable in $(0, +\infty)$.

We can therefore change the order of integration in (7) and write

$$\int_0^\infty \left[\frac{d^{m+z}}{dt^{m+z}} f(t)\right] e^{-pt} dt$$

$$= \frac{1}{\Gamma(1-z)} \int_0^\infty f^{(m+1)}(\xi) \left[\int_\xi^\infty (t-\xi)^{-z} e^{-pt} dt\right] d\xi$$

$$= \frac{1}{\Gamma(1-z)} \left[f^{(m)}(\xi) \int_\xi^\infty (t-\xi)^{-z} e^{-pt} dt\right]_0^\infty$$

$$- \int_0^\infty f^{(m)}(\xi) \frac{d}{d\xi} \left[\frac{1}{\Gamma(1-z)} \int_\xi^\infty (t-\xi)^{-z} e^{-pt} dt\right] d\xi$$

$$= p^{z-1} f^{(m)}(0) - \int_0^\infty f^{(m)}(\xi) \frac{d}{d\xi} \left[\frac{1}{\Gamma(1-z)} \int_\xi^\infty (t-e)^{-z} e^{-pt} dt\right] d\xi$$

$$= p^{z-1} f^{(m)}(0) - \int_0^\infty \frac{1}{\Gamma(1-z)} f^{(m)}(\xi) \frac{d}{d\xi} \left\{ \left[\frac{(t-\xi)^{1-z} e^{-pt}}{1-z}\right]_\xi^\infty \right.$$

$$\left. + \frac{p}{1-z} \int_\xi^\infty (t-\xi)^{1-z} e^{-pt} dt \right\} d\xi$$

$$= p^{z-1} f^{(m)}(0) + \int_0^\infty \frac{p}{\Gamma(1-z)} f^{(m)}(\xi) \left[\int_\xi^\infty (t-\xi)^{-z} e^{-pt} dt\right] d\xi$$

$$= p^{z-1} f^{(m)}(0) + \int_0^\infty f^{(m)}(\xi) p^z e^{-p\xi} d\xi$$

$$= p^z \left\{ p^{-1} f^{(m)}(0) - f^{(m-1)}(0) - p f^{(m-2)}(0) - \ldots - p^{m-1} f(0) + p^m \int_0^\infty f(\xi) e^{-p\xi} d\xi \right\}.$$

In the present paper we need to consider the case when $f^{(i)}(0)=0$, $(i=0, 1, \ldots, m)$; formula (6) is then

$$\int_0^\infty \left[\frac{d^{m+z}}{dt^{m+z}} f(t)\right] e^{-pt} dt = p^{m+z} \int_0^\infty f(t) e^{-pt} dt.$$

2. Dissipation in a plane wave

In the simple case of a plane wave, assuming $f=\eta\delta(z-z_0)$, the stress–strain relation (3) gives the following equation of motion

$$\rho \frac{\partial^2 u}{\partial t^2} + \mu \frac{\partial^2 u}{\partial x^2} + \eta \frac{\partial^{z_0}}{\partial t^{z_0}} \frac{\partial^2 u}{\partial x^2} = 0. \tag{8}$$

By taking the Laplace transform of equation (8) we have, assuming that

$$[u]_{t=0} = \left[\frac{\partial u}{\partial t}\right]_{t=0} = 0,$$

$$\rho p^2 U + \mu \frac{\partial^2 U}{\partial x^2} + \eta p^{z_0} \frac{\partial^2 U}{\partial x^2} = 0,$$

where U is the Laplace transform of u, and the nature of the motion depends on the roots of the following equation:

$$\eta \alpha^2 p^{z_0} + \rho p^2 + \mu \alpha^2 = 0. \tag{9}$$

The approximate solution of equation (9), neglecting the term in η, which we assume to be small in comparison with μ, is

$$p_0{}^2 = -\frac{\mu \alpha^2}{\rho}.$$

The solution which takes into account the dissipation is

$$p = i|p_0| \left\{ 1 + \frac{\eta |p_0|^{z_0} \alpha^2}{2|p_0|^2 \rho} \left(\cos \frac{\pi z_0}{2} + i \sin \frac{\pi z_0}{2} \right) \right\}$$

and the specific dissipation is

$$Q^{-1} = \frac{\eta |p_0|^{z_0}}{\mu} \sin \frac{\pi z_0}{2}.$$

3. Solution of the equations of dissipative elastodynamics

Let us first consider the equations which govern the motions of dissipative elastodynamic gravitating media as they result from the generalized stress–strain relations (3), and, to consider the problem from a more general point of view, let us also introduce a more general inertial law defined by the operator

$$o_3 = \rho \int_{a_3}^{b_3} f_3(r, z) \frac{\partial^z}{\partial t^z} dz,$$

which is the ordinary inertial law if

$$f_3(r,z)=\delta(2-z),\quad a_3\begin{cases}0\leqslant a_i<b_i<1\\ 1<a_3<b_3=2\end{cases}$$

The above-mentioned equations are then

$$\left.\begin{aligned}
F_1&\equiv o_1\frac{\partial\Delta}{\partial r}+\frac{2o_2}{r\sin\theta}\left(\frac{\partial\omega_\theta}{\partial\phi}-\frac{\partial\omega_\phi\sin\theta}{\partial\theta}\right)+\dot{o}_1\Delta+2\dot{o}_2\left(\frac{\partial s_1}{\partial r}-\Delta\right)\\
&\qquad+o_3u-\rho\frac{\partial V_0}{\partial r}\Delta+\rho\frac{\partial(V-V_0)}{\partial r}+\rho\frac{\partial}{\partial r}\left(s_1\frac{\partial V_0}{\partial r}\right)=0,\\
F_2&\equiv r^{-1}\left\{o_1\frac{\partial\Delta}{\partial\theta}+2o_2\left(\frac{\partial r\omega_\phi}{\partial r}-\frac{1}{\sin\theta}\frac{\partial\omega_r}{\partial\phi}\right)\right\}+\dot{o}_2\left(\frac{1}{r}\frac{\partial s_1}{\partial\theta}-\frac{s_2}{r}+\frac{\partial s_2}{\partial r}\right)\\
&\qquad+o_3s_2+\frac{\rho}{r}\frac{\partial(V-V_0)}{\partial\theta}+\frac{\rho}{r}\frac{\partial}{\partial\theta}\left(s_1\frac{\partial V_0}{\partial r}\right)=0,\\
F_3&\equiv r^{-1}\left\{o_1\frac{1}{\sin\theta}\frac{\partial\Delta}{\partial\phi}+2o_2\left(\frac{\partial\omega_r}{\partial\theta}-\frac{\partial r\omega_\theta}{\partial r}\right)\right\}+\dot{o}_2\left(\frac{1}{r\sin\theta}\frac{\partial s_1}{\partial\phi}+\frac{\partial s_3}{\partial r}-\frac{s_3}{r}\right)\\
&\qquad+o_3s_3+\frac{\rho}{r\sin\theta}\frac{\partial(V-V_0)}{\partial\phi}+\frac{\rho}{r\sin\theta}\frac{\partial}{\partial\phi}\left(s_1\frac{\partial V_0}{\partial r}\right)=0,\\
F_4&\equiv\nabla^2(V-V_0)-4\pi G\left(\rho\Delta+s_1\frac{\partial\rho}{\partial r}\right)=0,\\
\Delta&=r^{-2}\frac{\partial}{\partial r}(r^2s_1)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(s_2\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial s_3}{\partial\phi},\\
\omega_r&=\frac{1}{2r\sin\theta}\left\{\frac{\partial}{\partial\theta}(s_3\sin\theta)-\frac{\partial s_2}{\partial\phi}\right\},\\
\omega_\theta&=\frac{1}{2r\sin\theta}\left\{\frac{\partial s_1}{\partial\theta}-\frac{\partial}{\partial r}(rs_3\sin\theta)\right\},\\
\omega_\phi&=\frac{1}{2r}\left\{\frac{\partial rs_2}{\partial r}-\frac{\partial rs_1}{\partial\theta}\right\},
\end{aligned}\right\}\tag{10}$$

where $s=(s_1, s_2, s_3)$ is the displacement vector in spherical polar co-ordinates (r, θ, ϕ) and the s_i are functions of time, V_0 is the gravitational potential and $V-V_0$ is the perturbation of V_0 arising from the perturbation of the density field and from the attraction of the density perturbation of the deformed interfaces, and where

$$\left.\begin{aligned}
o_1&=\int_{a_1}^{b_1}f_1(r,z)\frac{\partial^z}{\partial t^z}dz+2\int_{a_2}^{b_2}f_2(r,z)\frac{\partial^z}{\partial t^z}dz,\\
\dot{o}_1&=\frac{\partial}{\partial r}o_1,\\
o_2&=\int_{a_2}^{b_2}f_2(r,z)\frac{\partial^z}{\partial t^z}dz,\quad \dot{o}_2=\frac{\partial}{\partial r}o_2.
\end{aligned}\right\}\tag{11}$$

To find a class of solution of equation (10) let us take its Laplace transform. The integration in equation (11) can be exchanged with the integration of the Laplace transform; we can assume

$$\left[\frac{\partial^{(m)} s_i}{\partial t^m}\right]_{t=0}=0, \quad m=0,\ 1.$$

Then

$$S_i O_1=\int_0^\infty o_1 s_i e^{-pt} dt=\left[\int_{a_1}^{b_1} p^z f_1(r, z) dz\right] \int_0^\infty s_i e^{-pt} dt+2\left[\int_{a_2}^{b_2} p^z f_2(r, z) dz\right] \int_0^\infty s_i e^{-pt} dt,$$

$$S_i O_2=\int_0^\infty p_2 s_i e^{-pt} dt=\left[\int_{a_2}^{b_2} p^z f_2(r, z) dz\right] \int_0^\infty s_i e^{-pt} dt,$$

$$S_i O_3=\int_0^\infty o_3 s_i e^{-pt} dt=\left[\rho \int_{a_3}^{b_3} f_3(r, z) \frac{\partial^z}{\partial t^z} dz\right] \int_0^\infty s_i e^{-pt} dt,$$

$$\dot{O}_i=\frac{\partial}{\partial r} O_i.$$

Let us write the following equations which are a consequence of equation (10):

$$\left.\begin{aligned}
&\int_w F_1 Y_n^k d\omega=0,\\
&\int_w\left[F_3 \frac{1}{\sin\theta} \frac{\partial Y_n^k}{\partial \phi}+F_2 \frac{\partial Y_n^k}{\partial \theta}\right] d\omega=0,\\
&\int_w\left[F_2 \frac{1}{\sin\theta} \frac{\partial Y_n^k}{\partial \phi}-F_3 \frac{\partial Y_n^k}{\partial \theta}\right] d\omega=0,\\
&\int_w F_4 Y_n^k d\omega=0;
\end{aligned}\right\} \tag{12}$$

where

$$\int_w L d\omega=\int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta \int_0^\infty L e^{-pt} dt,$$

$$Y_n^k=\begin{cases}
\left(\dfrac{2n+1}{4\pi}\right)^{\frac{1}{2}} P_n(\cos\theta) & \text{if } k=0\\[2ex]
\left[\dfrac{2n+1}{2\pi} \dfrac{(n-k)!}{(n+k)!}\right]^{\frac{1}{2}} P_n^{(k)}(\cos\theta) \cos k\phi & \text{if } k=1,2,\ldots,n\\[2ex]
\left[\dfrac{2n+1}{2\pi} \dfrac{(2n-k)!}{k!}\right]^{\frac{1}{2}} P_n^{(k-n)}(\cos\theta) \cos(k-n)\phi & \\
& \text{if } k=n+1,\ldots,2n.
\end{cases}$$

and set

$$\left.\begin{aligned}
R_{1,n}&=\int_w s_1\, Y_n^k\, d\omega,\\
R_{2,n}&=\int_w \left[s_2 \frac{\partial Y_n^k}{\partial\theta}+s_3\frac{\partial Y_n^k}{\partial\phi}\frac{1}{\sin\theta}\right] d\omega,\\
R_{3,n}&=\int_w \left[s_2 \frac{1}{\sin\theta}\frac{\partial Y_n^k}{\partial\phi}-s_3\frac{\partial Y_n^k}{\partial\theta}\right] d\omega,\\
(P_n-P_{0,n})&=\int (V-V_0)\, Y_n^k\, d\omega.
\end{aligned}\right\} \quad (13)$$

Performing the integration in equation (12), we see that if the vector $R\ (R_{1,n}, R_{2,n}, R_{3,n})$ and $P_n-P_{0,n}$ are solutions of the obtained system (14):

$$\left.\begin{aligned}
&\frac{d}{dr}[O_1\overline{\overline{\nabla}}]+2O_2\frac{d\overline{\overline{\nabla}}}{dr}-O_2\left[\frac{R_1}{r}n(n+1)-\frac{1}{r^2}\frac{d(rR_2)}{dr}n(n+1)\right]+2\dot{O}_1\frac{dR_1}{dr}\\
&\qquad -\rho\frac{dP_0}{dr}\overline{\overline{\nabla}}+\frac{d(P-P_0)}{dr}+\rho\frac{d}{dr}\left(R_1\frac{dP_0}{dr}\right)+O_3R_1=0,\\
&r^{-1}\left\{O_1\overline{\overline{\nabla}}-O_2\left[\frac{dR_1}{dr}-\frac{d^2(rR_2)}{dr^2}\right]+\dot{O}_2 r\left[\frac{R_1}{r}+r\frac{dR_2/r}{dr}\right]\right\}\\
&\qquad +\frac{d}{r}(P-P_0)+\rho\frac{R_1}{r}\frac{dP_0}{dr}+O_3R_2=0,\\
&\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d(P-P_0)}{dr}\right]-4\pi\rho G\overline{\overline{\nabla}}+\frac{n(n+1)}{r}\,\frac{P-P_0}{r}-4\pi G\rho\frac{d}{dr}R_1=0,\\
&O_2\left[\frac{1}{r}\frac{d^2rR_3}{dr^2}-\frac{n(n+1)}{r^2}R_3\right]+\dot{O}_2 r\frac{dR_3/r}{dr}+O_3R_3=0,
\end{aligned}\right\} \quad (14)$$

where

$$\overline{\overline{\nabla}}=-\frac{n(n+1)}{r}R_2+\frac{1}{r^2}\frac{d}{dr}(r^2R_1).$$

then they determine the Fourier co-ordinates of the Laplace transforms of $S\ (S_1, S_2, S_3)$ and $P-P_0$ of the vectors $s\ (s_1, s_2, s_3)$ and of the perturbation $V-V_0$. In fact from equation (13) it can be shown that, extending the procedure used by Caputo (1963), we can formally write:

$$\left.\begin{aligned}
S_1&=\sum_n\sum_k R_1\, Y_n^k,\\
S_2&=\sum_n\sum_k\left(R_2\frac{\partial Y_n^k}{\partial\theta}+R_3\frac{1}{\sin\theta}\frac{\partial Y_n^k}{\partial\phi}\right),\\
S_3&=\sum_n\sum_k\left(R_2\frac{1}{\sin\theta}\frac{\partial Y_n^k}{\partial\phi}-R_3\frac{\partial Y_n^k}{\partial\theta}\right),\\
\Psi&=\Psi_0+\sum_n\sum_k(P-P_0)Y_n^k.
\end{aligned}\right\} \quad (15)$$

4. Dissipation in the torsional oscillations of a shell

We shall discuss some models of dissipation, obtained by specializing the functions $f_i(r, z)$ appearing in equation (3), in the free purely torsional vibrations of a shell.

The equation which governs the motion of the torsional modes in an anelastic shell of radii r_1 and r_2, assuming a stress–strain relation of the type (3), is:

$$\int_{a_2}^{b_2} f_2(r,z)\frac{d^2}{dt^2}\left[\frac{1}{r}\frac{\partial^2 rs}{\partial r^2}+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\sin\theta}{\partial\theta}\right)\right] + \int_{a_2}^{b_2}\left[\frac{\partial}{\partial r}f_2(r,z)\right]\frac{\partial^z}{\partial t^z}\left[r\frac{\partial}{\partial r}\left(\frac{s}{r}\right)\right]dz=\rho\frac{\partial^2 s}{\partial t^2}. \quad (16)$$

We assume an Earth model defined by a liquid core and a homogeneous mantle, and assume also that the dissipation of energy due to the viscous interaction between the core and the mantle is negligible (Caputo 1966a), and that $\frac{\partial}{\partial r}f_2(r,z)=0$, then the boundary condition is

$$\begin{vmatrix} F_n(r_2) & F_{-n}(r_2) \\ F_n(r_1) & F_{-n}(r_1) \end{vmatrix}=0, \quad (17)$$

$$F_{\pm n}(r)=\frac{\partial}{\partial r}r^{-3/2}J_{\pm(n+\frac{1}{2})}(r\alpha),$$

where

$$S_3=\{A_n J_{n+\frac{1}{2}}(\alpha r)+(-1)^{n+1}A_{-n}J_{-n-\frac{1}{2}}(\alpha r)\}r^{-\frac{1}{2}}\frac{dP_n}{d\theta},$$

$$\alpha^{-2}=\rho^{-1}\int_{a_2}^{b_2} f_2(z)p^z\,dz,$$

is a solution of the Laplace transform of equation (16). $J_{n+\frac{1}{2}}(\alpha r)$ and $J_{-n-\frac{1}{2}}(\alpha r)$ are Bessel functions, and $P_n(\cos\theta)$ is a Legendre polynomial.

The solutions of equation (17) determine the periods of free oscillation and also the Q. Without loss of generality (4) can be written

$$f_2(z)=\mu+\int_{a_2}^{b_2}\bar{f}_2(z)\rho^2\,dz.$$

An interesting case arises when

$$\bar{f}_2(z)=\mu_1\delta(z-z_0+\varepsilon).$$

Equation (17) is then

$$\frac{\alpha^2}{\rho}(\mu+\mu_1 p^{z_0-\varepsilon})=-p^2,$$

which gives

$$p=i|p_0|\left\{1+\frac{\mu_1}{\mu}p_0{}^{z_0-\varepsilon}\right\}^{\frac{1}{2}},\quad |p_0|=\alpha\sqrt{\frac{\mu}{\rho}}.$$

If $z_0=2m$ (m integer) then we have

$$p=i|p_0|\left\{1+\frac{\mu_1|p_0|^{2m-\varepsilon}}{2\mu}\left[\cos\frac{\pi\varepsilon}{2}-i\sin\frac{\pi\varepsilon}{2}\right]\right\}^{\frac{1}{2}}$$

and if $m=0$

$$p=i|p_0|\left\{1+\frac{\mu_1|P_0|^{-\varepsilon}}{2\mu}\left[\cos\frac{\pi\varepsilon}{2}-i\sin\frac{\pi\varepsilon}{2}\right]\right\}.$$

The specific dissipation function is therefore

$$Q^{-1}=\frac{\mu_1}{\mu}|p_0|^{-\varepsilon}\left|\sin\frac{\pi\varepsilon}{2}\right|.$$

5. Analysis of experimental results: free spheroidal modes of the Earth and Rayleigh waves

A very extensive analysis of attenuation of Rayleigh waves has been made by Ben-Menahem (1965), who measured it from four great earthquakes by observation of multiple circuits around the Earth past one station. By Fourier analysis he obtained the specific attenuation factor in the period range between 300 and 40 s; from the attenuation factor he computed the specific dissipation.

The discrete spectrum of the spheroidal oscillation, where matter is both compressed and sheared, approaches the continuous Rayleigh wave spectrum. In order to extend the range where the Q is known we computed it also from the free spheroidal oscillations of periods between 5 and 15 min which followed the great Chilean earthquake. We used the 110 h record of this earthquake obtained at UCLA.

The record was subdivided into intervals of 54 h length. The initial points of these intervals were at $4{\cdot}5n$ and $13{\cdot}5n$ hours (n an integer) from the beginning; from the power spectrum analysis of these five intervals we obtained the decrease in the energy and subsequently Q by means of formula (24) of Caputo (1966a).

The scatter of the values in this frequency range is due to many causes. We believe that one of the principal causes is the number of aftershocks which followed the first main one. It seems that most of the energy of these aftershocks is introduced in the low frequencies; as a matter of fact in our analysis the Q's of the lowest frequencies appeared negative; they were, however, uniformly positive above the frequency of $1{\cdot}24$ Mc/s (the $l=7$, $n=0$ mode).

In Fig. 1 we report the data obtained by Ben-Menahem for Rayleigh waves and those which we obtained for the spheroidal oscillations of the Earth. The continuity of the two sets of data at their intersection is remarkable.

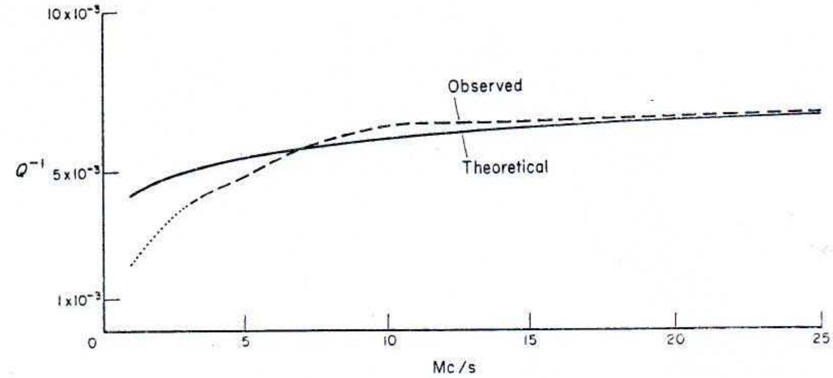

FIG. 1. Specific dissipation of Rayleigh waves (dashed line) and spheroidal oscillations (dotted line).

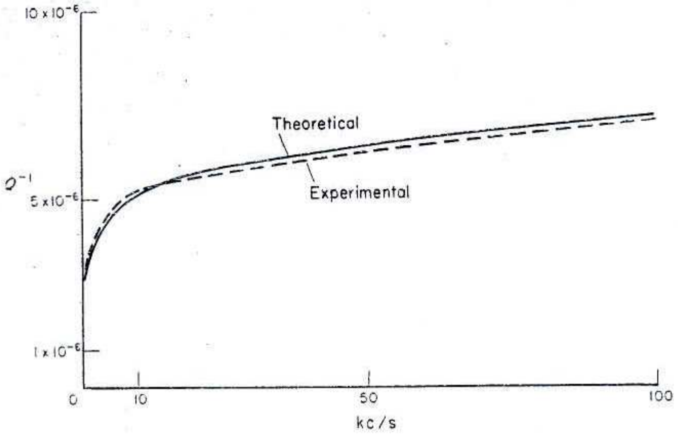

FIG. 2. Specific dissipation of longitudinal waves in an aluminium rod.

In the same figure one can see the theoretical curve obtained by setting $\varepsilon = -0{\cdot}15$ and $2\mu_1/(\lambda+2\mu) = 3{\cdot}7 \times 10^{-2}$ in the following formula which we obtained for the torsional oscillations and which we assume to be valid also for the spheroidal modes.

$$Q^{-1} = \frac{2\mu_1}{\lambda_1 + 2\mu_1} |p_0|^{-\varepsilon} \left| \sin \frac{\pi}{2} \varepsilon \right| . \tag{18}$$

One can notice that the argument between the theoretical curve and the observed values is satisfactory at the high frequencies although the observed values are slightly higher than the theoretical values. At low frequencies (below 7 Mc/s) the observed values are lower than the theoretical values. This could be explained as follows.

We know that the waves of longer periods sample the Earth at greater depth than those of shorter periods; therefore the waves of longer periods sample materials which are in different physical condition (e.g. greater pressure and temperature) than those sampled by waves of shorter periods. The Q^{-1} for various Earth's models obtained by Anderson & Archambeau (1964) all increase with depth to about 60 km and then they decrease; the theoretical curve for Q shown in Fig. 1 is in agreement with this behaviour.

6. Analysis of experimental results: longitudinal waves in an aluminium rod

Another check on this theory can be made using the Q's obtained by Zemanek & Rudnick (1961) for longitudinal waves in an aluminium rod in the period range between 10^{-5} and 10^{-3} s. These data are reproduced in Fig. 2. In Fig. 2 one can see a smoothed curve for Q^{-1} obtained from the data above, and also a theoretical curve obtained from a formula analogous to equation (18) in which we assumed again $\varepsilon = -0{\cdot}15$; the value of the amplitude factor in formula (18) is such as to obtain the best fit to the experimental values.

7. Conclusion

The elastic properties of solids are described by many parameters; according to this theory it seems also that the dissipative properties of dissipating solids depend on many parameters (i.e. ε, λ_1, and μ_1) which, in turn, depend on the physical conditions of the material.

This research is only a tentative investigation on how these parameters enter the dissipation law and on the analytic form of this law. No discussion is given of the physical interpretation and significance of the analytic law given; the law is empirical, as most physical laws are.

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1967 *May*.

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